Tracing the Supermode in Optical Waveguides

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Abstract: We derive in detail the eigensolutions – the "Supermodes" – of a coupled optical waveguide system, and present the corresponding numerical results. Based on the scheme of "supermode control," we suggest their applications in modern photonics.

The "Supermode" that we are going to talk about is anything but an alias for the Swedish DJs and producers Axwell and Steve Angello who have remixed Bronski Beat's "Smalltown Boy" and "Why," producing "Tell Me Why," in the spring of 2006. In what follows our supermodes, however, refer to the eigenmodes of a system of two (or even more) coupled optical waveguides. The initial study of these modes was carried on to understand the wave behaviors in semiconductor laser arrays [1]. In this article we first derive, step by step, the supermodes of a coupled optical waveguiding structure, then show the numerical simulation results, and finally suggest several potential applications in the area of optoelectronics.

We start out with the Maxwell's equations

$$\begin{cases} \nabla\times\vec{H} = \dfrac{\partial}{\partial t}\left(\varepsilon_0\vec{E}+\vec{P}\right) \\ \nabla\times\vec{E} = -\dfrac{\partial}{\partial t}\left(\mu_0\vec{H}+\vec{M}\right) = -\dfrac{\partial}{\partial t}\left(\mu\vec{H}\right) \end{cases} \tag{1}$$

Taking the curl of the second equation of (1) and using the first equation lead to

$$\begin{aligned} \nabla\times\left(\nabla\times\vec{E}\right) &= -\frac{\partial}{\partial t}\left(\mu\nabla\times\vec{H}\right) = -\mu\frac{\partial^2}{\partial t^2}\left(\varepsilon_0\vec{E}+\vec{P}\right) \\ &= -\mu\frac{\partial^2}{\partial t^2}\left(\varepsilon_0\vec{E}+\vec{P}_0+\vec{P}_{pert}\right) = -\mu\frac{\partial^2}{\partial t^2}\left(\varepsilon\vec{E}+\vec{P}_{pert}\right) \end{aligned} \tag{2}$$

Using vector identity and $\nabla\cdot\vec{E}=0$,

$$\nabla\times\left(\nabla\times\vec{E}\right) = \nabla\left(\nabla\cdot\vec{E}\right) - \nabla^2\vec{E} = -\nabla^2\vec{E} \tag{3}$$

we have

$$\nabla^2\vec{E} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} = \mu\frac{\partial^2\vec{P}_{pert}}{\partial t^2} \tag{4}$$

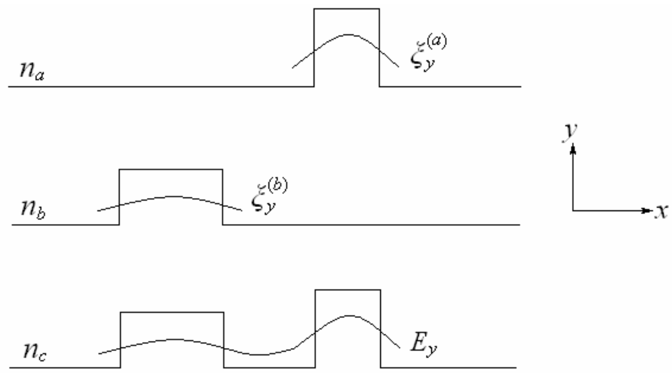


Fig. 1

Without loss of generality, we consider the case of two uncoupled 1-D (slab) waveguides with refractive index distributions $n_a(x)$ and $n_b(x)$ (See Fig. 1). The normalized transverse eigenmodes of each waveguide are $\xi_y^{(a)}(x)$ and $\xi_y^{(b)}(x)$, and their propagation constants are β_a and β_b. The modal field of the coupled guide structure with an index distribution $n_c(x)$ can be expressed as a superposition of the uncoupled fields

$$E_y(x,z) = A(z)\xi_y^{(a)}(x)e^{i(\omega t-\beta_a z)} + B(z)\xi_y^{(b)}(x)e^{i(\omega t-\beta_b z)} \tag{5}$$

where $\xi_y^{(a)}(x)$ and $\xi_y^{(b)}(x)$ satisfy

$$\begin{aligned} \left(\frac{\partial^2}{\partial x^2} - \beta_a^2\right)\xi_y^{(a)}(x) + \omega^2\mu\varepsilon_0 n_a^2(x)\xi_y^{(a)}(x) = 0 \\ \left(\frac{\partial^2}{\partial x^2} - \beta_b^2\right)\xi_y^{(b)}(x) + \omega^2\mu\varepsilon_0 n_b^2(x)\xi_y^{(b)}(x) = 0 \end{aligned} \tag{6}$$

Substitution of (5) into (4) yields

$$e^{i\omega t}\left[-2i\left(\beta_a\frac{\partial A}{\partial z}\xi_y^{(a)}e^{-i\beta_a z} + \beta_b\frac{\partial B}{\partial z}\xi_y^{(b)}e^{-i\beta_b z}\right)\right] = \mu\frac{\partial^2 \vec{P}_{pert}}{\partial t^2} \tag{7}$$

where we have used (6) and the "slow-varying" approximations $\left|\frac{\partial^2 A}{\partial z^2}\right| \ll \beta_a\left|\frac{\partial A}{\partial z}\right|$ and $\left|\frac{\partial^2 B}{\partial z^2}\right| \ll \beta_b\left|\frac{\partial B}{\partial z}\right|$. The perturbation polarization is

$$\begin{aligned} P_{pert}(\vec{r},t) &= P(\vec{r},t) - P_0(\vec{r},t) = \varepsilon_0 \varepsilon_r^{coup} E(\vec{r},t) - \varepsilon_0 \varepsilon_r^{orig} E(\vec{r},t) \\ &= \varepsilon_0 n_c^2(x)\left[A(z)\xi_y^{(a)}(x)e^{i(\omega t-\beta_a z)} + B(z)\xi_y^{(b)}(x)e^{i(\omega t-\beta_b z)} \right] \\ &\quad - \varepsilon_0 n_a^2(x)A(z)\xi_y^{(a)}(x)e^{i(\omega t-\beta_a z)} - \varepsilon_0 n_b^2(x)B(z)\xi_y^{(b)}(x)e^{i(\omega t-\beta_b z)} \\ &= e^{i\omega t}\varepsilon_0\left[A(z)\xi_y^{(a)}(x)\left(n_c^2(x)-n_a^2(x)\right)e^{-i\beta_a z} + B(z)\xi_y^{(b)}(x)\left(n_c^2(x)-n_b^2(x)\right)e^{-i\beta_b z} \right] \end{aligned} \tag{8}$$

Plugging (8) into (7), multiplying both sides by $\xi_y^{(a)}(x)$, then integrating over all x lead to

$$\begin{aligned} -4i\omega\frac{dA}{dz}e^{-i\beta_a z} = \varepsilon_0\frac{\partial^2}{\partial t^2}\Big[&\int_{-\infty}^{\infty}\left(\xi_y^{(a)}(x)\right)^2\left(n_c^2(x)-n_a^2(x)\right)dx A(z)e^{-i\beta_a z} \\ &+\int_{-\infty}^{\infty}\xi_y^{(a)}(x)\xi_y^{(b)}(x)\left(n_c^2(x)-n_b^2(x)\right)dx B(z)e^{-i\beta_b z}\Big] \end{aligned} \tag{9}$$

where we have used the mode orthogonality condition

$$\int_{-\infty}^{\infty}\xi_y^{(a)}(x)\xi_y^{(b)}(x)dx = \frac{2\omega\mu}{\beta_a}\delta_{a,b} \tag{10}$$

Likewise,

$$\begin{aligned} -4i\omega\frac{dB}{dz}e^{-i\beta_b z} = \varepsilon_0\frac{\partial^2}{\partial t^2}\Big[&\int_{-\infty}^{\infty}\xi_y^{(a)}(x)\xi_y^{(b)}(x)\left(n_c^2(x)-n_a^2(x)\right)dx A(z)e^{-i\beta_a z} \\ &+\int_{-\infty}^{\infty}\left(\xi_y^{(b)}(x)\right)^2\left(n_c^2(x)-n_b^2(x)\right)dx B(z)e^{-i\beta_b z}\Big] \end{aligned} \tag{11}$$

If we introduce the definitions

$$\begin{aligned} \kappa_{ab,ba} &= \frac{\omega\varepsilon_0}{4}\int_{-\infty}^{\infty}\left[n_c^2(x)-n_{b,a}^2(x)\right]\xi^{(a)}(x)\xi^{(b)}(x)dx \\ M_{a,b} &= \frac{\omega\varepsilon_0}{4}\int_{-\infty}^{\infty}\left[n_c^2(x)-n_{a,b}^2(x)\right]\left(\xi^{(a,b)}(x)\right)^2 dx \end{aligned} \tag{12}$$

then we have

$$\begin{cases} \dfrac{dA}{dz} = -i\kappa_{ab}Be^{-i(\beta_b-\beta_a)z} - iM_a A \\ \dfrac{dB}{dz} = -i\kappa_{ba}Ae^{-i(\beta_a-\beta_b)z} - iM_b B \end{cases} \tag{13}$$

To simplify (13), we introduce $A=\tilde{A}e^{-iM_a z}$ and $B=\tilde{B}e^{-iM_b z}$ and define $\beta'_{a,b}=\beta_{a,b}+M_{a,b}$, then we get

$$E_y(x,z) = e^{i\omega t}\left(\tilde{A}e^{-i\beta'_a z}\xi_y^{(a)} + \tilde{B}e^{-i\beta'_b z}\xi_y^{(b)}\right) \tag{14}$$

and

$$\begin{cases} \dfrac{d\tilde{A}}{dz} = -i\kappa_{ab}\tilde{B}e^{-i(\beta_b'-\beta_a')z} \\ \dfrac{d\tilde{B}}{dz} = -i\kappa_{ba}\tilde{A}e^{i(\beta_b'-\beta_a')z} \end{cases} \tag{15}$$

Assuming $\kappa_{ab} = \kappa_{ba} = |\kappa|$ and defining $\kappa = -i\kappa_{ab} = -i|\kappa|$, (15) becomes a set of standard coupled-mode equations

$$\begin{cases} \dfrac{d\tilde{A}}{dz} = \kappa\tilde{B}e^{-i(\beta_b'-\beta_a')z} \\ \dfrac{d\tilde{B}}{dz} = -\kappa^*\tilde{A}e^{i(\beta_b'-\beta_a')z} \end{cases} \tag{16}$$

Eq. (14) can be expressed in the basis of $\left\{\xi_y^{(a)}, \xi_y^{(b)}\right\}$ as a column vector

$$\underline{E} = \begin{vmatrix} E^{(1)} \\ E^{(2)} \end{vmatrix} = \begin{vmatrix} \tilde{B}e^{-i\beta_b'z} \\ \tilde{A}e^{-i\beta_a'z} \end{vmatrix} \tag{17}$$

so that

$$\frac{d\underline{E}}{dz} = \begin{vmatrix} -i\beta_b' & -\kappa^* \\ \kappa & -i\beta_a' \end{vmatrix} \begin{vmatrix} E^{(1)} \\ E^{(2)} \end{vmatrix} = \underline{C}\underline{E} \tag{18}$$

A propagating supermode, *by its definition*, is a field solution whose *z* (the propagation direction) dependence is only through a propagation phase factor $e^{i\gamma z}$, i.e. $\underline{E}(z) = \underline{E}(0)e^{i\gamma z}$, so

$$\frac{d\underline{E}}{dz} = i\gamma\underline{E} \tag{19}$$

Combining (18) and (19),

$$\left(\underline{C} - i\gamma\underline{I}\right)\underline{E} = 0 \tag{20}$$

i.e.

$$\begin{vmatrix} -i\beta_b' - i\gamma & -\kappa^* \\ \kappa & -i\beta_a' - i\gamma \end{vmatrix} \begin{vmatrix} E^{(1)} \\ E^{(2)} \end{vmatrix} = 0 \tag{21}$$

To have nontrivial solutions, we require that

$$\begin{Vmatrix} -i\beta_b' - i\gamma & -\kappa^* \\ \kappa & -i\beta_a' - i\gamma \end{Vmatrix} = -\beta_b'\beta_a' - \gamma^2 - \gamma\left(\beta_a' + \beta_b'\right) + |\kappa|^2 = 0 \tag{22}$$

leading to

$$\gamma = -\frac{\beta_a' + \beta_b'}{2} \pm \frac{1}{2}\sqrt{\left(\beta_b' - \beta_a'\right)^2 + 4\left|\kappa\right|^2} \tag{23}$$

Let $\overline{\beta} = \frac{\beta_a' + \beta_b'}{2}$, $\delta = \frac{\beta_b' - \beta_a'}{2}$, and $S = \sqrt{\delta^2 + \left|\kappa\right|^2}$, then $\gamma_{1,2} = -\overline{\beta} \pm S$, and the corresponding eigenfields (supermodes) are

$$\underline{E}_1 = \begin{vmatrix} E_1^{(1)} \\ E_1^{(2)} \end{vmatrix} = \begin{vmatrix} \kappa^* \\ -i\beta_b' - i\gamma_1 \end{vmatrix} e^{i\gamma_1 z} = \begin{vmatrix} \frac{i\kappa^*}{\delta + S} \\ 1 \end{vmatrix} e^{-i\left(\overline{\beta} - S\right)z}$$
$$\underline{E}_2 = \begin{vmatrix} E_2^{(1)} \\ E_2^{(2)} \end{vmatrix} = \begin{vmatrix} \kappa^* \\ -i\beta_b' - i\gamma_2 \end{vmatrix} e^{i\gamma_2 z} = \begin{vmatrix} \frac{i\kappa^*}{\delta - S} \\ 1 \end{vmatrix} e^{-i\left(\overline{\beta} + S\right)z} \tag{24}$$

There are three limiting cases of our special interest:

(1) $\delta < 0, \left|\delta\right| \gg \left|\kappa\right|$,

so $S = \left|\delta\right|\left(1 + \left|\frac{\kappa}{\delta}\right|^2\right)^{\frac{1}{2}} \approx \left|\delta\right|\left(1 + \frac{1}{2}\left|\frac{\kappa}{\delta}\right|^2\right) = -\delta - \frac{\left|\kappa\right|^2}{2\delta}$ and

$$\underline{E}_1 = \begin{vmatrix} -\frac{2i\delta}{\kappa} \\ 1 \end{vmatrix} e^{-i\beta_b' z} = \begin{vmatrix} -\frac{1}{\varepsilon} \\ 1 \end{vmatrix} e^{-i\beta_b' z}$$
$$\underline{E}_2 = \begin{vmatrix} \frac{i\kappa^*}{2\delta} \\ 1 \end{vmatrix} e^{-i\beta_a' z} = \begin{vmatrix} \varepsilon \\ 1 \end{vmatrix} e^{-i\beta_a' z} \tag{25}$$

where $\varepsilon \equiv \frac{\left|\kappa\right|}{2\left|\delta\right|}$.

(2) $\delta = 0$,

so $S = \left|\kappa\right|$ and

$$\underline{E}_1 = \begin{vmatrix} \frac{i\kappa^*}{\left|\kappa\right|} \\ 1 \end{vmatrix} e^{-i\left(\overline{\beta} - \left|\kappa\right|\right)z} = \begin{vmatrix} -1 \\ 1 \end{vmatrix} e^{-i\beta_o z}$$
$$\underline{E}_2 = \begin{vmatrix} -\frac{i\kappa^*}{\left|\kappa\right|} \\ 1 \end{vmatrix} e^{-i\left(\overline{\beta} + \left|\kappa\right|\right)z} = \begin{vmatrix} 1 \\ 1 \end{vmatrix} e^{-i\beta_e z} \tag{26}$$

(3) $\delta > 0, \delta \gg |\kappa|$,

so $S = \delta\left(1+\frac{|\kappa|^2}{\delta^2}\right)^{\frac{1}{2}} \approx \delta + \frac{|\kappa|^2}{2\delta}$ and

$$\begin{aligned} \underline{E}_1 &= \begin{vmatrix} \frac{i\kappa^*}{2\delta} \\ 1 \end{vmatrix} e^{-i\beta_a' z} = \begin{vmatrix} -\varepsilon \\ 1 \end{vmatrix} e^{-i\beta_a' z} \\ \underline{E}_2 &= \begin{vmatrix} -\frac{2i\delta}{\kappa} \\ 1 \end{vmatrix} e^{-i\beta_b' z} = \begin{vmatrix} \frac{1}{\varepsilon} \\ 1 \end{vmatrix} e^{-i\beta_b' z} \end{aligned} \tag{27}$$

The three cases are illustrated in Fig. 2.

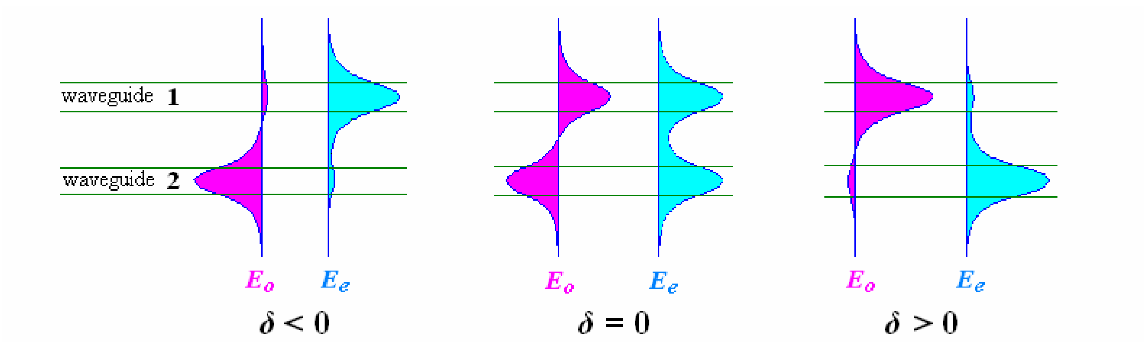
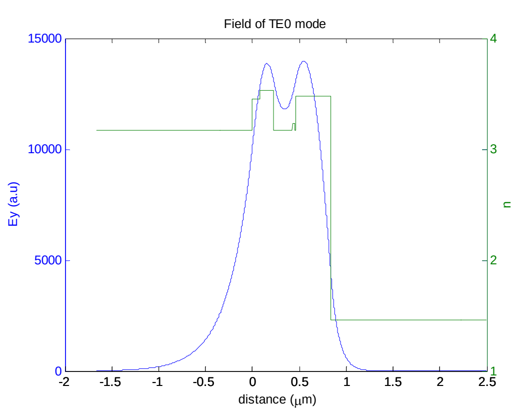


Fig. 2

Now that we understand the supermode behaviors in a coupled waveguide system, we simulate the eigenmodes of a multilayer dielectric structure consisting of two high refractive index slabs. The results are shown in Fig. 3. Both of them correspond to the synchronism point, where the phase mismatch parameter $\delta \rightarrow 0$, and the energy is roughly half and half concentrated in each slab.



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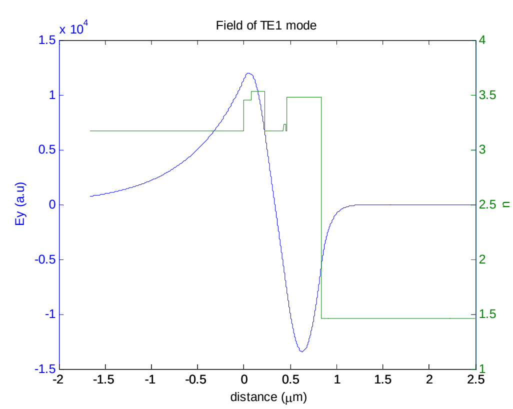

Odd

Fig. 3

The idea of supermode was initially introduced by Kapon *et al* to analyze the modes propagating in the semiconductor laser arrays [1] and the formalism then formed into shape in Yariv's book [2]. With the rapid development of micro-fabrication and material processing techniques, people now are able to make almost any kind of waveguides with desired dimensions. We have mentioned that different set of parameters δ and κ results in different supermode shape, hence different ratio of fractions of energy concentrated in the constituent guides. The control of δ and κ can be realized in practice by designing waveguide configurations, say, the layer width, refractive index structure, etc. As the mode propagates in the structure, its mode shape, as well as the energy distribution, experiences transformation as the composite guide structure gradually changes. Based on the scheme of "mode control," we can thus make full use of each waveguide's strong suit so as to design a variety of hybrid waveguide structures with different optical functions, like laser resonators, optical amplifiers, and modulators [3].

References:

1. E. Kapon, J. Katz, and A. Yariv, "Supermode Analysis of Phase-Locked Arrays of Semiconductor-Lasers," Opt. Lett. **9**, 125-127 (1984).

2. A. Yariv, *Optical Electronics in Modern Communications* (Oxford Univ. Press, New York, 1997).

3. A. Yariv, and X. K. Sun, "Supermode Si/III-V hybrid lasers, optical amplifiers and modulators: A proposal and analysis," Opt. Express **15**, 9147-9151 (2007).